Topology Course Notes (KSM1C03)

Day 7 : 29th August, 2025

`product of connected spaces -- interval connected`

7.1 Connectedness (cont.)

Theorem 7.1: (Product of connected spaces is connected)

Suppose $\{X_\alpha\}_{\alpha\in I}$ is a collection of connected spaces. Let $X = \Pi X_\alpha$ be the product space. Then, X is connected.

Proof

For finite product $X \times Y$, fix a point $y_0 \in Y$, and observe that

$$X \times Y = \bigcup_{x\in X} \Big(\underbrace{\{x\} \times Y \cup X \times \{y_0\}}_{C_x} \Big) .$$

Note that C_x is connected since it is the union of two connected sets $\{x\} \times Y \cong Y$ and $X \times \{y_0\} \cong X$ (check!), and they have a common point (x, y_0). But then $X \times Y$ is connected, as $\bigcap_{x\in X} C_x = X \times \{y_0\} \neq \emptyset$. This can be generalized to any finite product.

As for the infinite product, fix a point $z = (z_\alpha) \in X$ (*If you don't want to assume axiom of choice, then X could be empty, which is still a connected set*). Consider the subset

$$A := \{(x_\alpha) \in X \mid \text{all but finitely many } x_\alpha = z_\alpha\} .$$

Since $X = \bar{A}$, it is enough to show that A is connected. Firstly, for any finite $J \subset I$, define

$$A_J := \{(x_\alpha) \in X \mid x_\alpha = z_\alpha \text{ for any } \alpha \in I \setminus J\} .$$

Observe that $A_J \cong \Pi_{\alpha\in J} X_\alpha$ (check!), and hence, connected. Next, observe that $A = \bigcup_{\text{J} \subset \text{I finite}} A_J$, and $\bigcap_{\text{J} \subset \text{I finite}} A_J = \{z\}$. Thus, A is connected as well. But then $X = \bar{A}$ is connected. □

Exercise 7.2: (Box topology may not be connected)

Consider $X = \mathbb{R}^{\mathbb{N}}$ equipped with the box topology, where $\mathbb{R}$ has the usual topology. Check that the following sets are nontrivial clopen sets of X.

a) $U_0 := \{(x_n) \mid \lim x_n = 0 \text{ in } \mathbb{R}\}$.

b) $U_1 := \{(x_n) \mid \{x_n\}$ is a bounded sequence in $\mathbb{R}$ $\}$.

Theorem 7.3: (Closed interval is connected)

The closed interval $[a, b] \subset \mathbb{R}$ for some $a < b$ is connected.

Proof

Suppose $[a, b] = A \sqcup B$ for some open (and hence closed) nontrivial subsets $\emptyset \subsetneq A, B \subset [a, b]$. Without loss of generality, assume that $a \in A$. Consider the set

$$C := \{c \mid [a, c] \subset A\}.$$

Observe that since $a \in C$ since $\{a\} = [a, a] \subset A$. Clearly b is an upper bound for C. Then, there is a least upper bound, say, $L := \sup C$.

As A is open, there is some $\epsilon_0 > 0$ such that $[a, a + \epsilon_0) \subset A$, and thus $L \geq a + \epsilon_0 > a$. Let us show that $L \in C$. Firstly, note that for any $0 < \epsilon \leq \epsilon_0$, we have some $L - \epsilon \leq c_0 \in C$, and thus, $[a, L - \epsilon] \subset [a, c_0] \subset A$. In other words, $(L - \epsilon, L + \epsilon) \cap A \neq \emptyset$. But then, L is a closure point of A (in the subspace topology of $[0, 1]$). Since A is closed, we have $L \in A$. As A is open as well, we have some $\epsilon_1 \leq \epsilon_0$ such that $(L - \epsilon_1, L + \epsilon_1) \cap [a, b] \subset A$. But then,

$$[a, L] = [a, L - \epsilon_1] \cup (L - \epsilon_1, L] \subset A,$$

which shows that $L \in C$.

Now, $L \leq b$, as b is an upper bound of C. If possible, suppose $L < b$. Then, for some $\epsilon > 0$ small, we have $[L - \epsilon, L + \epsilon] \subset [a, b]$. Choosing ϵ smaller, and using the openness of A, we have $[a, L + \epsilon] = [a, L] \cup (L, L + \epsilon] \subset A$, which implies $L + \epsilon \in C$, contradicting $L = \sup C$. Hence, $L = b$. But then, $[a, L] = [a, b] \subset A$, contradicting that $B \neq \emptyset$.

Thus, $[a, b]$ is connected. □

Proposition 7.4: (All intervals are connected)

Any finite or infinite interval, whether open, closed or semi-open, of $\mathbb{R}$ is connected. In particular, $\mathbb{R}$ is connected

Proof

Let us show that $\mathbb{R}$ is connected. If not, then $\mathbb{R} = U \sqcup V$ is a separation by open sets. Pick some $a \in U$ and $b \in V$. Then, $[a, b] = ([a, b] \cap U) \sqcup ([a, b] \cap V)$ is a separation of $[a, b]$. This is a contradiction as $[a, b]$ is connected. Hence, $\mathbb{R}$ is connected.
Similar argument works for the other cases. □

Exercise 7.5: (Intermediate value property)

Suppose $f : [a, b] \to \mathbb{R}$ is a continuous map. If $f(a) < f(b)$, then for any $f(a) < x < f(b)$ there exists some $a < c < b$ such that $f(c) = x$.