Topology Course Notes (KSM1C03)

Day 6 : 27th August, 2025

`connectedness -- components`

6.1 Connectedness

Definition 6.1: (Connected space)

A space X is called *connected* if the only clopen sets (i.e., simultaneously open and closed sets) of X are $\emptyset$ and X itself. If there is a nontrivial clopen set $\emptyset \subsetneq U \subsetneq X$, then X is called *disconnected*.

Proposition 6.2: (Disconnected space)

For a space X, the following are equivalent.

1) X is disconnected.

2) X can be written as the disjoint union of two open sets $X = U \sqcup V$, such that, $\emptyset \subsetneq U \subsetneq X$ and $\emptyset \subsetneq V \subsetneq X$.

3) X can be written as the disjoint union of two closed sets $X = F \sqcup G$, such that, $\emptyset \subsetneq F \subsetneq X$ and $\emptyset \subsetneq G \subsetneq X$.

4) There is a surjective continuous map $X \to \{0, 1\}$, where $\{0, 1\}$ is given the discrete topology.

Proof

The equivalence of $1, 2, 3$ follows from the definition. Suppose $f : X \to \{0, 1\}$ is a surjective continuous map. Then, X can be written as the disjoint union $X = f^{-1}(0) \sqcup f^{-1}(1)$, each of which are non-trivial open sets. Conversely, if $X = U \sqcup V$ for some nontrivial open sets, then $f : X \to \{0, 1\}$ defined by $f(U) = 0$ and $f(V) = 1$ is a surjective continuous map. □

Theorem 6.3: (Image of connected set)

Suppose $f : X \to Y$ is a continuous map. Then, for any connected $A \subset X$, we have $f(A) \subset Y$ is connected. In particular, if X is connected, then so is $f(X)$.

Proof

Suppose $f(A) \subset Y$ is disconnected. Then, there is a surjective continuous map $g : f(A) \to \{0, 1\}$. But then, $h := g \circ f : A \to \{0, 1\}$ is a surjective continuous map, a contradiction. Hence, $f(A)$ is connected. □

Definition 6.4: (Connected component)

Given $x \in X$, the *connected component* of X containing x is the largest possible connected subset containing x.

Proposition 6.5: (Existence of connected component)

Given $x \in X$, the connected component of X containing X is defined as the

$$\mathcal{C}(x) := \bigcup \{A \mid x \in A \subset X, A \text{ is connected}\}.$$

Proof

Observe that $\{x\}$ is a connected set, and hence, the family is non-empty. Let us check $\mathcal{C}(x)$ is connected. If not, then there exists open sets $U, V \subset X$ such that

- $\emptyset \subsetneq \mathcal{C}(x) \cap U \subsetneq \mathcal{C}(x)$,
- $\emptyset \subsetneq \mathcal{C}(x) \cap V \subsetneq \mathcal{C}(x)$, and
- $\mathcal{C}(x) = (\mathcal{C}(x) \cap U) \sqcup (\mathcal{C}(x) \cap V)$.

Now, for any connected set A containing x, we have

$$A = (A \cap U) \sqcup (A \cap V).$$

Then, both

$$\emptyset \subsetneq A \cap U \subsetneq A, \quad \text{and} \quad \emptyset \subsetneq A \cap V \subsetneq A$$

cannot appear simultaneously. Hence, either $A \subset U$ or $A \subset V$. Thus, we can define the two collections

$$\mathcal{U} := \{A \mid x \in A \subset X,\ A \text{ is connected},\ A \subset U\}, \mathcal{V} := \{A \mid x \in A \subset X,\ A \text{ is connected},\ A \subset V\}.$$

Since $x \in A$ for all such A, we must have either $\mathcal{U} = \emptyset$ or $\mathcal{V} = \emptyset$. Without loss of generality, assume $\mathcal{V} = \emptyset$. But then, $\mathcal{C}(x) \cap V = \emptyset$, a contradiction. Hence, $\mathcal{C}(x)$ is connected. By construction, it is the largest such connected set which contains x. Thus, $\mathcal{C}(x)$ is the connected component containing x. □

Exercise 6.6: (Hyperbola and axes)

Suppose

$$A = \{(x, y) \mid xy = 1\} \cup \{(x, y) \mid xy = 0\} \subset \mathbb{R}.$$

Show that A has three connected components.

Theorem 6.7: (Closure is connected)

If $A \subset X$ is a connected set, then for any subset B satisfying $A \subset B \subset \bar{A}$, we have B is connected. In particular, $\bar{A}$ is connected.

Proof

Suppose, we have $B = U \sqcup V$ for some open sets $\emptyset \subsetneq U, V \subsetneq B$. Since $A \subset B$, we have $A \subset U$ or $A \subset V$ (otherwise, $A = (A \cap U) \sqcup (A \cap V)$ will be a separation of A). Without loss of generality, say, $A \subset U \Rightarrow \bar{A}^B \subset \bar{U}^B$. Now, $U \subset B$ is closed (in B), as $B \backslash U = V$ is open (in B). In particular, $\bar{U}^B = U$. On the other hand, $\bar{A}^B = \bar{A} \cap B \supset B \Rightarrow B \subset \bar{A}^B \subset \bar{U}^B = U$. This contradicts that $\emptyset \subsetneq V \subsetneq B$. Hence, B is connected. □

Example 6.8: (Discrete space)

In a discrete space X, every singleton $\{x\}$ is a connected component. Any subset with at least two elements is then disconnected.

Definition 6.9: (Totally disconnected space)

A space X is called *totally disconnected* if the only connected components of x are precisely the singletons.

Note that totally disconnected spaces need not be discrete.