Class Test 1 (Solution)

26th August, 2025

Q1. Let $X = \mathbb{R}/\mathbb{Q}$ be the identification space, i.e, the quotient space induced by the relation $a \sim b$ if and only if $a, b \in \mathbb{Q}$ or $a = b \in \mathbb{R} \setminus \mathbb{Q}$. Let $q : \mathbb{R} \to X$ be the quotient map.

a) Describe the open sets $U \subset \mathbb{R}$ which are q-saturated (i.e, $U = q^{-1}(q(U))$).

b) What is the closure of the equivalence class $[x] \in X$ for any $x \in \mathbb{R} \setminus \mathbb{Q}$?

c) What is the closure of the equivalence class $[0] \in X$?

d) Determine (with brief explanation) whether X is T_2, T_1, or T_0.

a) Clearly, $\emptyset$ is open and q-saturated. Suppose U is a *nonempty* open set, which is also q-saturated. Since U is open, and $U \neq \emptyset$, there is some rational $x \in U$. Then,

$$[x] \in q(U) \Rightarrow \mathbb{Q} \subset q^{-1}(q(U)) = U.$$

So, any nonempty q-saturated open set must contain $\mathbb{Q}$. Conversely, suppose U is any (nonempty) open set, with $\mathbb{Q} \subset U$. Then,

$$q(U) = \{[0]\} \cup \{[x] \mid x \in U \setminus \mathbb{Q}\} \Rightarrow q^{-1}(q(U)) = \mathbb{Q} \cup (U \setminus \mathbb{Q}) = U.$$

So, U is then q-saturated.

Thus, the q-saturated open sets of $\mathbb{R}$ are precisely the emptyset, and any open set containing $\mathbb{Q}$.

b) For any $x \in \mathbb{R} \setminus \mathbb{Q}$, we have

$$q^{-1}(X \setminus \{[x]\}) = \mathbb{R} \setminus \{x\},$$

which is open in $\mathbb{R}$. Then, by the definition of quotient topology, $X \setminus \{[x]\}$ is open in X. Thus, $\{[x]\}$ is closed, and $\overline{\{[x]\}} = \{[x]\}$.

c) Let $C \subset X$ be a closed set with $[0] \in C$. Then, $q^{-1}(C) \subset \mathbb{R}$ is closed, and $\mathbb{Q} \subset q^{-1}(C)$. But then,

$$\mathbb{R} = \overline{\mathbb{Q}} \subset \overline{q^{-1}(C)} = q^{-1}(C).$$

So, $q^{-1}(C) = \mathbb{R}$. This implies $C = X$. Thus, $\overline{\{[0]\}} = X$, as the closure is the smallest closed set containing $[0]$.

d) Clearly X is *not* T_1 (and hence *not* T_2) as $\{[0]\}$ is not a closed set [by c)]. For any $[x] \neq [y]$, we can assume that $[y] \neq [0]$. Then, $U = X \setminus \{[y]\}$ is an open set [by b)], with $[x] \in U$ and $[y] \notin U$. Thus, X is T_0.

Q2. Let X be an infinite set, and fix a point $p \in X$. Consider the collection

$$\mathcal{T}_p := \{S \subset X \mid p \in S\} \cup \{\emptyset\}.$$

a) Verify that $\mathcal{T}_p$ is a topology on X (called the *particular point topology*).

b) Consider a sequence $\{x_n\}$ in X, whose tail (i.e, the subsequence $\{x_n\}_{n \geq N}$ for some $N \geq 1$) looks like

$$x, p, x, p, x, p, \ldots.$$

Show that x_n converges to x. If $x \neq p$, then show that the sequence does not converge to p.

c) Determine (with brief explanation) whether $(X, \mathcal{T}_p)$ is T_2, T_1, or T_0.

a) Given $\emptyset \in \mathcal{T}_p$. Also, $X \in \mathcal{T}_p$ as $p \in X$. For any $\{U_\alpha \in \mathcal{T}_p\}$, we have $p \in \cup U_\alpha$, and so $\mathcal{T}_p$ is closed under arbitrary union. For any $\{U_i \in \mathcal{T}_p\}_{i=1}^n$, we have $p \in \cap_{i=1}^n U_i$, and so $\mathcal{T}_p$ is closed under finite intersection. Thus, $\mathcal{T}_p$ is a topology on X.

b) Consider any open neighborhood $x \in U$. Then, $p \in U$ as well. But then a tail of the sequence is contained in $\{x, p\} \subset U$. So, $x_n \to x$. For $x \neq p$, consider $U = \{p\}$ which is an open neighborhood of p. Then, for any $N \geq 1$, there is always some $n \geq N$ such that $x_n = x \notin U$. So, $x_n \not\to p$.

c) Any two (nonempty) open sets in $\mathcal{T}_p$ contains p in the intersection, so X cannot be T_2.

For the point p, observe that $\overline{\{p\}} = X$. Indeed, for any $x \neq p$, any open set containing x must contain p, and thus, x is a closure point of $\{p\}$. So, X is not T_1.

For any $x \neq y$, we can assume that $y \neq p$. Then, $U = X \setminus \{y\}$ is an open set containing x, but not containing y. So, X is T_0.