Topology Course Notes (KSM1C03)

Day 5 : 21th August, 2025

`Hausdorff axiom --` T_2, T_1, T_0 `-- convergence of sequence -- sequential continuity -- quotient space`

5.1 Hausdorff Axiom

Definition 5.1: (Hausdorff space)

A space X is called *Hausdorff* (or a *T_2-space*) if for any $x, y \in X$ with $x \neq y$, there exists open neighborhoods $x \in U_x \subset X, y \in U_y \subset X$, such that $U_x \cap U_y = \emptyset$. In other words, any two points of a Hausdorff space can be separated by open sets.

Exercise 5.2: (Product of T_2-spaces)

Suppose $\{X_\alpha\}$ is a collection of T_2-spaces. Show that $X = \Pi X_\alpha$ is T_2 with respect to the product topology (and hence, with respect to the box topology as well).

Being Hausdorff is a very desirable property of a space.

Exercise 5.3: (Metric spaces are Hausdorff)

If (X, d) is a metric space, then show that the metric topology is Hausdorff.

Proposition 5.4: (Points are closed in Hausdorff space)

Suppose X is a Hausdorff space. Then, $\{x\}$ is a closed subset of X for any $x \in X$.

Proof

Suppose $y \neq x$. Then, by Hausdorff property, we have some open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. In particular, y is not a closure point of $\{x\}$. Thus, $\{x\}$ is closed. □

Note that in the proof, the full strength of the Hausdorff property is not used.

Definition 5.5: (T_1 space)

A space X is called a *T_1-space* (or a *Fréchet space*) if for any $x \in X$, the subset $\{x\}$ is a closed set.

Exercise 5.6: (T_1 but not T_2 space)

Given an example of a space X which is T_1 but not T_2.

Exercise 5.7: (T_1-space equivalent definition)

Let X be a space. Show that the following are equivalent.

a) X is a T_1 space.

b) For any $x, y \in X$ with $x \neq y$, there exists open neighborhoods $x \in U_x \subset X$ and $y \in U_y \subset X$ such that $y \notin U_x$ and $x \notin U_y$.

c) Any $A \subset X$ is the intersection of all open sets containing A.

d) For any $A \subset X$ and $x \in X$, we have x is a limit point of A if and only every open neighborhood of x contains infinitely many points of A. (What happens when X is finite?!)

Definition 5.8: (T_0-space)

A space X is called a *T_0-space* (or a *Kolmogorov space*) if for any two points $x \neq y \in X$, there exists an open set $U \subset X$ which contains exactly one of x and y.

Remark 5.9: (Topolgoically distinguishable and separable)

Suppose $x, y \in X$ are two points. Note the following hierarchy.

- **(Distinct)** If $x \neq y$, we say x, y are distinct.
- **(Topologically distinguishable)** If there is at least one open set that contains exactly one of x and y, we say x, y are topologically distinguishable.
- **(Separable)** If there are two neighborhoods U_x, U_y of x, y respectively, which does not contain the other, we say x, y are topologically separable.
- **(Separated by opens)** If there are two neighborhoods U_x, U_y of x, y respectively, such that $U_x \cap U_y = \emptyset$, we say x, y are separated by open sets.

Later, we shall see how this continues to points and closed sets as well.

Exercise 5.10: (T_0 but not T_1 space)

Given an example of a space X which is T_0 but not T_1. What about

Exercise 5.11: (Zariski topology)

Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Give it the topology $\mathcal{T} = \{\emptyset, \mathbb{F}^\times, \mathbb{F}\}$, where $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. Consider the family of polynomial functions $\mathcal{F} := \{p : \mathbb{F}^n \to \mathbb{F}\}$. The topology induced by $\mathcal{F}$ on $\mathbb{F}^n$ is known as the *Zariski topology*. Determine whether it is T_0, T_1 or T_2.

5.2 Convergence of sequence

Definition 5.12: (Convergence of sequence)

Suppose $\{x_n\}_{n\geq 1}$ is a sequence of points in a space X (i.e, $x : \mathbb{N} \to X$ is a function). We say $\{x_n\}$ converges to a limit $x \in X$ if for any open neighborhood U of x, there is a natural number N_U such that $x_n \in U$ for all $n \geq N_U$.

Exercise 5.13: (Convergence in metric)

Check that the notion of convergence in a metric space is equivalent to the usual notion (i.e, $x_n \to x$ if and only if $d(x_n, x) \to 0$). In particular, they are the same from real analysis.

Example 5.14

Suppose X is an indiscrete space, with at least two distinct points $x, y \in X$. Consider the sequence

$$x_n = \begin{cases} x, & n \text{ is odd}, \\ y, & n \text{ is even}. \end{cases}$$

Observe that the sequence converges to both x and y. In fact, any sequence in X converges to every point in the space X. Note that an indiscrete space is not even T_0.

Example 5.15

Suppose $X = \{0, 1\}$, with topology $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$. This space $(X, \mathcal{T})$ is known as *Sierpiński space*. Clearly it is T_0, but not T_1 since $\{0\}$ is not closed. Now, consider the sequence $x_n = 0$ for all $n \geq 1$. Then, $\{x_n\}$ converges to both 0 and 1.

Proposition 5.16: (Convergence in T_2)

Suppose $\{x_n\}$ is a sequence in a T_2-space X. Then, $\{x_n\}$ can converge to at most one point in X.

Proof

If possible, suppose $\{x_n\}$ converges to distinct point $x \neq y$. By Hausdorff property, we have two open neighborhoods U_x, U_y of x, y respectively, such that $U_x \cap U_y = \emptyset$. We also have two natural numbers N_1, N_2 such that $x_n \in U_x$ for all $n \geq N_1$ and $x_n \in U_y$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then,

$$x_n \in U_x \cap U_y, \quad \text{for all } n \geq N.$$

This is a contradiction. Thus, any sequence can converge to at most one point. □

5.3 Sequential Continuity

Definition 5.17: (Sequenttial continuity)

A function $f : X \to Y$ is said to be *sequentially continuous* if for any converging sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.

Proposition 5.18: (Continuous functions are sequentially continuous)

Suppose $f : X \to Y$ is a continuous map. Then f is sequentially continuous.

Proof

Suppose $x_n \to x$ is a converging sequence in X. Let $f(x) \in U \subset Y$ be an arbitrary open neighborhood. Then, it follows from continuity of f that $f^{-1}(U) \subset X$ is open. Clearly $x \in f^{-1}(U)$. Hence, there is some $N \geq 1$ such that $x_n \in f^{-1}(U)$ for all $n \geq N$. This implies $f(x_n) \in U$ for all $n \geq N$. Since U was arbitrary, we see that $f(x_n) \to f(x)$. But this means f is sequentially convergent. □

Proposition 5.19: (Sequential continuity in metric spaces)

Suppose (X, d) is a metric space with the metric topology, and Y be any space. Then, any sequentially continuous map $f : X \to Y$ is a continuous map.

Proof

Let $U \subset Y$ be open. In order to show $f^{-1}(U) \subset X$ is open, we show that any $x \in f^{-1}(U)$ is an interior point of $f^{-1}(U)$. Consider the metric balls $B_n := B_d\left(x, \frac{1}{n}\right) \subset X$. If possible, suppose $B_n \not\subset f^{-1}(U)$ for any n. Pick points $x_n \in f^{-1}(U) \setminus B_n$, and observe that $x_n \to x$ (Check!). Then, we have $f(x_n) \to f(x)$. Since U is an open neighborhood of $f(x)$, we have some $N \geq 1$ such that $f(x_n) \in U$ for all $n \geq N$. But then $x_n \in f^{-1}(U)$ for $n \geq N$, which is a contradiction. Hence, we must have that for some $N_0 \geq 1$ the metric ball $B_{N_0} \subset f^{-1}(U)$. Thus, x is an interior point. Since x is arbitrary, we get $f^{-1}(U)$ is open. Consequently, f is continuous. □

Caution 5.20: (Sequential conitinuity may not imply continuity)

In general, sequential continuity may not imply continuity! Consider X to be a space equipped with the cocountable topology. Then, any convergent sequence in X is eventually constant. That is, if $x_n \to x$ in X, then for some $N \geq 1$, we have $x_n = x$ for all $n \geq N$. But then any function $f : X \to Y$ is sequentially continuous (Why?). Assume X is uncountable, so that the cocountable topology is not the discrete topology. Then, there are non-continuous maps on X. For example, consider $Y = X$ equipped with the discrete topology, and then look at the identity map $\mathrm{Id} : X \to Y$.

5.4 Quotient space

Definition 5.21: (Quotient map)

Given a space $(X, \mathcal{T})$ and a function $f : X \to Y$ to a set Y, the *quotient topology* on Y is defined as

$$\mathcal{T}_f := \left\{ U \mid f^{-1}(U) \in \mathcal{T} \right\}.$$

The map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_f)$ is called a *quotient map*. In other words, f is a quotient map if $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

Proposition 5.22: (Quotient topology is topology)

The quotient topology $\mathcal{T}_f$ is indeed a topology on Y, and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_f)$ is continuous.

Proof

We check the axioms.

i) $\emptyset \in \mathcal{T}_f$ since $\emptyset = f^{-1}(\emptyset) \in \mathcal{T}$.

ii) $Y \in \mathcal{T}_f$ since $X = f^{-1}(Y) \in \mathcal{T}$.

iii) For any collection $\{U_\alpha \in \mathcal{T}_f\}$, we have $f^{-1}(\bigcup U_\alpha) = \bigcup f^{-1}(U_\alpha) \in \mathcal{T}$. Thus, $\mathcal{T}_f$ is cloes under arbitrary union.

iv) For a finite collection $\{U_i\}_{i=1}^k$, we have $f^{-1}\left(\bigcap U_i\right) = \bigcap f^{-1}(U_i) \in \mathcal{T}$. Thus, $\mathcal{T}_f$ is closed under finite intersection.

Hence, $\mathcal{T}_f$ is a topology. By construction, f is then continuous. □

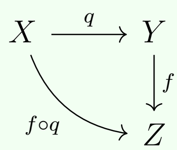
Theorem 5.23: (Universal property of quotient topology)

Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are given. Then, for any set function, $q : X \to Y$, the following are equivalent.

1. $\mathcal{T}_Y$ is the quotient topology induced by q (in other words, q is a quotient map).

2. $\mathcal{T}_Y$ is the finest (i.e, largest) topology for which q is continuous.

3. $\mathcal{T}_Y$ is the unique topology having the following property :



for any space $(Z, \mathcal{T}_Z)$ and any set map $f : Y \to Z$, we have f is continuous if and only if $f \circ q$ is continuous

Proof

Suppose q is a quotient map. If possible, there is some topology $\mathcal{S}_Y$ on Y such that $\mathcal{T}_Y \subsetneq \mathcal{S}_Y$ and $q : (X, \mathcal{T}_X) \to (Y, \mathcal{S}_Y)$ is continuous. Since $\mathcal{S}_Y$ is strictly finer than $\mathcal{T}_Y$, there is some set $U \in \mathcal{S}_Y \setminus \mathcal{T}_Y$. But then $q^{-1}(U) \in \mathcal{T}_X$, as q is continuous. This implies $U \in calT_Y$, a contradiction. Hence, the quotient topology is the finest topology on Y making q continuous.

Conversely, suppose $\mathcal{T}_Y$ is the finest topology so that q is continuous. Recall the quotient topology is

$$\mathcal{T}_q = \{U \mid q^{-1}(U) \in \mathcal{T}_X\}$$

Since q is continuous, for each $U \in \mathcal{T}_Y$ we have $q^{-1}(U) \in \mathcal{T}_X$. In particular, $\mathcal{T}_Y \subset \mathcal{T}_q$. Also, $q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_q)$ is continuous. Since $\mathcal{T}_Y$ is the finest such topology, we must have $\mathcal{T}_Y = \mathcal{T}_q$.

Next, suppose $\mathcal{T}_Y$ is the quotient topology. Let us choose some space $(Z, \mathcal{T}_Z)$ and set map $f : Y \to Z$. If f is continuous, then we have $f^{-1}(U) \in \mathcal{T}_Y$ for all $U \in \mathcal{T}_Z$. Then,

$$(f \circ q)^{-1}(U) = q^{-1}\left(f^{-1}(U)\right) \in \mathcal{T}_X,$$

by the definition of quotient topology. Thus, $f \circ q$ is continuous. On the other hand, suppose $f \circ q$ is continuous. Then, for any $U \circ \mathcal{T}_Z$, we have $q^{-1}\left(f^{-1}(U)\right) \in \mathcal{T}_X$. But then again by the definition of quotient topology, we have $f^{-1}(U) \in \mathcal{T}_Y$, which shows that f is continuous. Thus, $\mathcal{T}_Y$ satisfies the property. If possible, suppose $\mathcal{S}_Y$ is another topology on Y satisfying the property. Let us take $Z = (Y, \mathcal{T}_Y)$ and $f = \mathrm{Id}_Y : (Y, \mathcal{S}_Y) \to (Y, \mathcal{T}_Y)$. Then, we have f is continuous if and only if $f \circ q$ is continuous. But, $f \circ q = q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, which is continuous being the quotient map. Hence, f is continuous. This implies $\mathcal{T}_Y \subset \mathcal{S}_Y$. But $\mathcal{T}_Y$ is the finest topology for which q is continuous, and hence, $\mathcal{T}_Y = \mathcal{S}_Y$. This proves the uniqueness.

Finally, suppose $\mathcal{T}_Y$ is the unique topology satisfying the property above. We show that the quotient topology $\mathcal{T}_q$ satisfies the property. Suppose $(Z, \mathcal{T}_Z)$ is some space, and $f : Y \to Z$ is a set map. If $f : (Y, \mathcal{T}_q) \to (Z, \mathcal{T}_Z)$ is continuous, then for any $U \in \mathcal{T}_Z$ we have

$$(f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U)) \in \mathcal{T}_X,$$

since $f^{-1}(U) \in \mathcal{T}_q$. On the other hand, if $f \circ q$ is continuous, then for any $U \in \mathcal{T}_Z$ we have $q^{-1}\left(f^{-1}(U)\right) \in \mathcal{T}_X$, which implies, $f^{-1}(U) \in \mathcal{T}_q$. Thus, f is continuous. In particular, $\mathcal{T}_q$ satisfies the property, and hence, $\mathcal{T}_Y$ is the quotient topology by uniqueness.

This concludes the proof. □

Remark 5.24: (Quotient map and surjectivity)

Suppose $f : X \to Y$ is a quotient map. Assume that f is *not* surjective. Then, for any $y \in Y \setminus f(X)$ we have $f^{-1}(y) = \emptyset \subset X$ open, and hence, $\{y\}$ is open in Y. In other words, $Y \setminus f(X)$ has the discrete topology. Also, $f(X) \subset Y$ is both an open and closed set. Hence, the open and closed sets of $f(X)$ in the subspace topology are precisely the same in the actual (quotient) topology on Y. For these reasons, we can (and usually we do) assume that a quotient map is surjective.

Remark 5.25: (Surjective map and equivalence relation)

Suppose $f : X \to Y$ is a surjective map. Then, the collection $\bigsqcup_{y \in Y} f^{-1}(y)$ is a partition on X, and hence, induces an equivalence relation. Indeed, we can define $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Conversely, given any equivalence relation $\sim$ on X, we see that $q : X \to X/_\sim$, is a surjective map, where $X/_\sim$ is the collection of all equivalence classes under the relation $\sim$.

Given a set map $f : X \to Y$, a subset $S \subset X$ is called *saturated* (or *f-saturated*) if $S = f^{-1}(f(S))$ holds.

Exercise 5.26: (Saturated open set)

Given a quotient map $q : X \to Y$, a set $U \subset X$ is q-saturated if and only if it is the union of the equivalence classes of its elements (i.e, $U = \bigcup_{x \in U} [x]$).

Definition 5.27: (Identification topology)

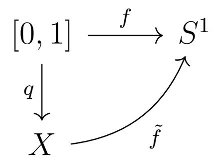
Given an equivalence relation $\sim$ on a space X, the *identification topology* on the set $Y = X/_\sim$ of all equivalence classes is the quotient topology induced by the map $q : X \to Y$, which sends $x \mapsto [x]$. The quotient map q is called the *identification map*.

Proposition 5.28: $[0,1]/0,1$ is S^1

Consider $\{0,1\} \subset [0,1]$, and let $X = [0,1]/_{\{0,1\}}$ be the identification space. Then, X is homeomorphic to the circle $S^1 := \{(x,y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$.

Proof

Consider the map $f : [0,1] \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. Clearly, f is continuous and surjective. Also, $f(0) = (1,0) = f(1)$.



Passing to the quotient $X = [0,1]/\{0,1\}$, we get a map $\tilde{f} : X \to S^1$ defined by $\tilde{f}([x]) = f(x)$. It is easy to see that $\tilde{f}$ is well-defined, and hence, by the property of the quotient topology, $\tilde{f}$ is continuous. Now, $\tilde{f}$ is surjective (as f was), and moreover, it is injective.
In order to show $\tilde{f}$ is open, we consider the two cases.

i) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0,1\} \notin V$. Then, $q^{-1}(V) \subset [0,1]$ is an open set, which is actually contained in $(0,1)$. In particular, $q^{-1}(V)$ is a union of open intervals. Observe that (by drawing picture or otherwise) f maps such open intervals to open arcs of S^1 (which are open in S^1). Then, $\tilde{f}(V) = f\left(q^{-1}(V)\right)$ is open.

ii) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0,1\} \in V$. Then, $q^{-1}(V)$ is the union of open intervals of $(0,1)$, as well as, $[0, \epsilon_1) \cup (1-\epsilon_2, 1]$ for some $\epsilon_1, \epsilon_2 > 0$. We have already seen that any open intervals get mapped to open arcs. Also, $f\left([0,\epsilon_1) \cup (1-\epsilon_2, 2]\right)$ is another open arc in S^1 containing the point $(0,1)$. Thus, $\tilde{f}(V) = f\left(q^{-1}(V)\right)$ is open in S^1.

Hence, $\tilde{f} : X \to S^1$ is a homeomorphism. □

Exercise 5.29: ($\mathbb{R}/\mathbb{Z}$ is S^1)

Consider the quotient space $X = \mathbb{R}/\mathbb{Z}$, where the equivalence relation is given as $a \sim b$ if and only $a - b \in \mathbb{Z}$. Show that X is homeomorphic to the circle S^1.