Topology Course Notes (KSM1C03)

Day 4 : 20^{th} August, 2025

`product spaces`

4.1 Product space

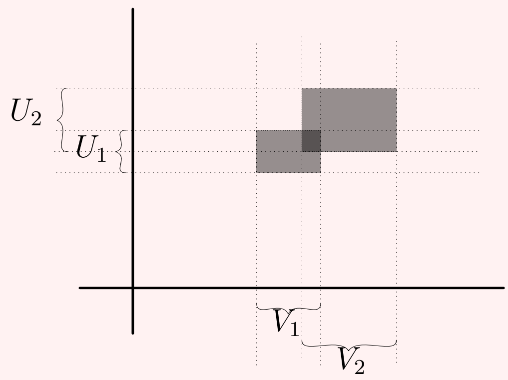
Definition 4.1: (Finite product)

Given $X_1, \ldots, X_n$, the *product space* is the Cartesian product $X = X_1 \times \cdots \times X_n$, equipped with the topology generated by the basis

$$\mathcal{B} := \{U_1 \times \cdots \times U_n \mid U_i \subset X_i \text{ is open for all } 1 \leq i \leq n\}.$$

Caution 4.2: (Product topology and basis)

Note that the product topology on $X \times Y$ is *generated by the basis* $\{U \times V \mid U \subset X, V \subset Y \text{ are open}\}$. In particular, not all open sets look like a product.



An open set $(U_1 \times V_1) \cup (U_2 \cup V_2)$

Exercise 4.3: (Finite product induced by projection)

Show that the product topology on $X := X_1 \times \cdots \times X_n$ is induced by the collection of projection maps $\{\pi_i : X \to X_i\}_{i=1}^n$.

Motivated by this, let us define the product of arbitrary many spaces.

Definition 4.4: (Product topology)

Let $\{X_\alpha\}_{\alpha \in \mathcal{I}}$ be an arbitrary collection topological spaces, indexed by the set $\mathcal{I}$. Denote the product as the set of tuples

$$X := \Pi_{\alpha\in\mathcal{I}} X_\alpha = \{(x_\alpha) \mid x_\alpha \in X_\alpha \text{ for all } \alpha \in \mathcal{I}\}.$$

Then, the *product topology* (or the *Tychonoff topology*) on X is defined as the topology induced by the collection of projection maps $\{\pi_\alpha : X \to X_\alpha\}_{\alpha\in\mathcal{I}}$

Proposition 4.5: (Product topology basis)

The product topology is generated by the basis

$$\mathcal{B} := \{\Pi_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open, and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in \mathcal{I}\}.$$

Proof

It is easy to see that $\mathcal{B}$ is a basis. Indeed, elements of $\mathcal{B}$ are of the form

$$\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{-1}(U_{\alpha_k}),$$

for some open sets $U_{\alpha_1} \subset X_{\alpha_1}, \ldots, U_{\alpha_k} \subset X_{\alpha_k}$. The claim follows. □

Definition 4.6: (Box topology)

Given a collection $\{X_\alpha\}$ of spaces, the *box topology* on $X = \Pi X_\alpha$ is generated by the subbasis

$$\mathcal{S} := \{\Pi U_\alpha \mid U_\alpha \subset X_\alpha \text{ is open}\}.$$

Clearly, the box topology is *finer* than the product topology. In particular, the projection maps are continuous with respect to the box topology as well.

Exercise 4.7: (Box and product topology)

Show that for a finite product $X_1 \times \cdots \times X_n$ of spaces, the box and the product topology agree. Moreover, show that for an infinite product, the box topology is always strictly finer than the product topology.

Caution 4.8: (Product topology always means Tychonoff topology)

Unless explicitly mentioned, always assume that a product space is given the Tychonoff topology. The box topology is usually too fine (i.e, has too many open sets), and is useful in constructing counter-examples.

Theorem 4.9: (Universal property of the product topology)

Let $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ be a collection of topological spaces. For a space $(Z, \mathcal{T})$, and a collection of continuous maps $g_\alpha : Z \to X_\alpha$, consider the following property.

P(Z, g_α) : Given a space Y and any collection of continuous maps $f_\alpha : Y \to X_\alpha$, there exists a unique continuous map $h : Y \to Z$, such that $f_\alpha = g_\alpha \circ h$.

Then, the following holds.

a) The product space $X = \Pi X_\alpha$ with the product topology, and the projection maps $\pi_\alpha : X \to X_\alpha$ satisfies the property $\mathsf{P}(X, \pi_\alpha)$

b) If (Z, g_α) is any other tuple satisfying the property $\mathsf{P}(Z, g_\alpha)$, then there is a homeomorphism $\Phi : Z \to X$ such that $\pi_\alpha \circ \Phi = g_\alpha$

Proof

Given any $f_\alpha : Y \to X_\alpha$, define $h : Y \to X = \Pi X_\alpha$ by

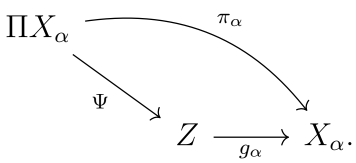
$$h(y) = (f_\alpha(y)),$$

which clearly satisfies $\pi_\alpha \circ h = f_\alpha$, and hence, is unique. Let us show h is continuous. We only need to check continuity for subbasic open sets, which are of the form $\pi_{\alpha_0}^{-1}(U)$ for some $U \subset X_{\alpha_0}$ open. Now,

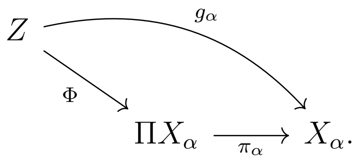
$$h^{-1}(\pi_{\alpha_0}(U)) = (\pi_{\alpha_0} \circ h)^{-1}(U) = f_{\alpha_0}^{-1}(U),$$

which is open as f_{α_0} is continuous. Thus, the property $\mathsf{P}(X, \pi_\alpha)$ holds.

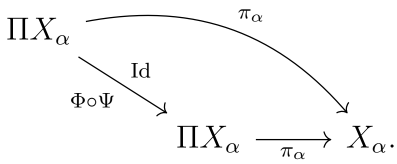
The second part is a standard diagram chasing argument. Suppose (Z, γ_α) is a tuple satisfying $\mathsf{P}(Z, \gamma_\alpha)$. Then, consider the collection of commutative diagrams.



The existence of (unique) $\Psi : \Pi X_\alpha \to Z$ is justified by $\mathsf{P}(Z, g_\alpha)$. Next, consider the collection of commutative diagrams



Again, existence of (unique) Φ is justified by $\mathsf{P}(\Pi X_\alpha, \pi_\alpha)$. Now, consider the following case.



Let us observe that

$$\pi_\alpha \circ (\Phi \circ \Psi) = (\pi_\alpha \circ \Phi) \circ \Psi = g_\alpha \circ \Psi = \pi_\alpha,$$

which follows from the previous two diagrams. Also, clearly

$$\pi_\alpha \circ \mathrm{Id} = \pi_\alpha.$$

Hence, by the *uniqueness* in $\mathsf{P}(\Pi X_\alpha, \pi_\alpha)$, we must have $\Phi \circ \Psi = \mathrm{Id}_{\Pi X_\alpha}$. By a similar argument, we get $\Psi \circ \Phi = \mathrm{Id}_Z$. Hence, Φ is a homeomorphism, with inverse given by Ψ. □

Exercise 4.10: (Map into box topology)

Suppose $X = \mathbb{R}^{\mathbb{N}}$, equipped with the box topology. Show that the map $f : \mathbb{R} \to X$ defined by $f(t) = (t, t, \dots)$ is not continuous.

Hint

Consider the open set $U = \Pi(-\frac{1}{n}, \frac{1}{n}) = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots \subset X$.