Topology Course Notes (KSM1C03)

Day 3 : 14th August, 2025

`closure -- interior -- boundary -- subspaces -- continuous function`

3.1 Limit points and closure

Definition 3.1: (Limit point)

Given a space X and a subset $A \subset X$, a point $x \in X$ is called a *limit point* (or *cluster point*, or *point of accumulation*) of A if for any open set $U \subset X$, with $x \in U$, we have $A \cap U$ contains a point other than x.

Exercise 3.2

Show that if A is a closed set of X, then A contains all of its limit points. Give an example of a space X and a subset $A \subset X$, such that

a) there is a limit point x of A which is not an element of A, and

b) there is an element $a \in A$ which is not a limit point of A.

Definition 3.3: (Adherent and isolated points)

Given a subset $A \subset X$, a point $x \in X$ is called an *adherent point* (or *points of closure*) if every open neighborhood of x intersects A. An adherent point which is *not* a limit point is called an *isolated point* of A (which is then necessarily an element of A).

Definition 3.4: (Closure of a set)

Given $A \subset X$, the *closure* of A, denoted $\bar{A}$ (or $\mathrm{cl}\, A$), is the smallest closed set of X that contains A.

Exercise 3.5

Show that $A \subset X$ is closed if and only if $A = \bar{A}$.

Exercise 3.6

For any $A \subset X$, show that $\bar{A}$ is the intersection of all closed sets of X containing A. In particular, $A \subset \bar{A}$.

Proposition 3.7

Given $A \subset X$, we have

$$\bar{A} = \{x \in X \mid x \text{ is an adherent point of } A\}.$$

Proof

Suppose $x \in X$ is an adherent point of A. Let $C \subset X$ be a closed set containing A. If possibly, say $x \notin C \Rightarrow x \in X \setminus C$. Now, $X \setminus C$ is an open set, and $A \cap (X \setminus C) = \emptyset$. This contradicts that x is an adherent point of A. Thus, $x \in C$. Since C was arbitrary, we get $x \in \bar{A}$. Thus, $\bar{A}$ contains all the adherent points of A.
Conversely, suppose $x \in \bar{A}$. If possible, suppose x is not an adherent point of A. Then, there exists some open set U such that $x \in U$ and $U \cap A = \emptyset$. Now, $A \subset (X \setminus U)$, and $X \setminus U$ is a closed set. So, $\bar{A} \subset X \setminus U \Rightarrow \bar{A} \cap U = \emptyset$. This means, $x \notin \bar{A}$, a contradiction. Thus, x must be an adherent point of A. This concludes the claim. □

Exercise 3.8

Suppose $A = \{x_n\} \subset \mathbb{R}$ is an infinite set.

a) If $x = \lim_n x_n$ exists, then show that x is a limit point of A.

b) If $x \in \mathbb{R}$ is a limit point of A, then show that there is a subsequence $\{x_{n_k}\}$ with $x = \lim_k x_{n_k}$.

Suppose,

$$x_n = \begin{cases} 1 - \frac{1}{k}, & n = 2k, \\ 2 + \frac{1}{k}, & n = 2k+1. \end{cases}$$

What are the limit points of $A = \{x_n \mid n \in \mathbb{N}\}$?

Definition 3.9: (Locally finite)

Given any collection $\mathcal{A}$ of subsets of a space X, we say $\mathcal{A}$ is a *locally finite* collection if for each $x \in X$, there exists an open neighborhood $x \in U$, such that U intersects only finitely many subsets from $\mathcal{A}$

Proposition 3.10: (Closure of locally finite collection)

Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha \in \mathcal{I}}$ is a locally finite collection of subsets of X. Then, $\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A_\alpha}$.

Proof

We only show $\overline{\bigcup_\alpha A_\alpha} \subset \bigcup_\alpha \overline{A_\alpha}$. If possible, suppose $x \in \overline{\bigcup_\alpha A_\alpha}$ and $x \notin \bigcup \overline{A_\alpha}$. By local finiteness, we have some open neighborhood U of x, which only intersects, say, $A_{\alpha_1}, \dots, A_{\alpha_n} \in \mathcal{A}$ (the list can be empty as well). Now, consider the set $V = U \setminus \bigcup_{i=1}^n \overline{A_{\alpha_i}}$, which is open (check). Clearly $x \in V$. But $V \cap (\bigcup A_\alpha)$. This contradicts the fact that x is a closure point. □

3.2 Interior

Definition 3.11: (Interior of a set)

Given $A \subset X$, the *interior* of A, denoted $\mathring{A}$ (or $\operatorname{int} A$), is the largest open set contained in A. A point $x \in \mathring{A}$ is called an *interior point* of A.

Exercise 3.12: (Interior of open sets)

For any $A \subset X$ show that $\mathring{A}$ is the union of all open sets contained in A. In particular, show that $A \subset X$ is open if and only if $A = \mathring{A}$.

Exercise 3.13: (Interior point)

Given $A \subset X$, show that a point $x \in X$ is an interior point of A if and only if there exists some open set $U \subset X$ such that $x \in U \subset A$.

3.3 Boundary

Definition 3.14: (Boudary of a set)

Given $A \subset X$, the *boundary* of A, denoted ∂A (or $\operatorname{bd} A$), is defined as

$$\partial A = \bar{A} \cap \overline{(X \setminus A)}.$$

Clearly boundary of any set is always a closed set. Also, observe the following. Given any $A \subset X$, a point $x \in X$ can satisfy exactly one of the following.

a) There exists an open set U with $x \in U \subset A$ (whence x is an interior point of A).

b) There exists an open set U with $x \in U \subset X \setminus A$ (whence x is an interior point of $X \setminus A$).

c) For any open set U with $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) = \emptyset$ (whence x is a boundary point of A).

Exercise 3.15

Given $A \subset X$, show that

$$\partial A = \{x \in X \mid \text{for any } U \subset X \text{ open, with } x \in U, \text{ we have } U \cap A \neq \emptyset \neq U \cap (X \setminus A)\}$$

Exercise 3.16

Find out the boundaries of A, when

a) $A = \{(x, y) \mid x^2 + y^2 < 1\} \subset \mathbb{R}^2$, and

b) $A = \{(x, y, z) \mid x^2 + y^2 < 1, \ z = 0\} \subset \mathbb{R}^3$.

Caution 3.17

The above exercise shows that our intuitive notion of boundary of a disc may be misleading! In order to justify our intuition that "the boundary of a disc is the circle", one needs to treat it as a 'manifold with boundary'.

3.4 Subspaces

Definition 3.18: (Subspace topology)

Given a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, the *subspace topology* on A is defined as the collection

$$\mathcal{T}_A \coloneqq \{U \subset A \mid U = A \cap O \text{ for some } O \in \mathcal{T}\}.$$

We say $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$.

Exercise 3.19

Suppose $U \subset X$ is an open set. What are the open subsets of U in the subspace topology? What are the closed sets?

Proposition 3.20: (Closure in subspace)

Let $Y \subset X$ be a subspace. Then, a subset of Y is closed in Y if and only if it is the intersection of Y with a closed set of X. Consequently, for any $A \subset Y$, the closure of A in the subspace topology is given as $\bar{A}^Y = \bar{A} \cap Y$.

Proof

For any $C \subset Y$, we have

$$\begin{aligned} C \text{ is closed in } Y &\Leftrightarrow Y \setminus C \text{ is open in } Y \text{ (by definition of closed set)} \\ &\Leftrightarrow Y \setminus C = Y \cap U, \text{ for some } U \subset X \text{ open (by definition of subspace topology).} \end{aligned}$$

Then,

$$C = Y \setminus (Y \setminus C) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap \underbrace{(X \setminus U)}_{\text{closed in } X}.$$

On the other hand, for any closed set $F \subset X$, we have

$$Y \setminus (Y \cap F) = Y \setminus F = Y \cap \underbrace{(X \setminus F)}_{\text{open in } X},$$

which implies $Y \setminus (Y \cap F)$ is open in F. But then $Y \cap F$ is closed in Y.

Now,

$$\bar{A}^Y = \bigcap_{\substack{C \subset Y \text{ closed} \\ A \subset C}} C = \bigcap_{\substack{C \subset X \text{ closed} \\ A \subset C}} (Y \cap C) = Y \cap \left(\bigcap_{\substack{C \subset X \text{ closed} \\ A \subset C}} C \right) = Y \cap \bar{A}.$$

This concludes the proof. $\square$

Exercise 3.21: (Interior and subspace)

Prove or disprove : Let $Y \subset X$ be a subspace, and $A \subset Y$. Then, the interior of A in Y (with respect the subspace topology) is $\mathring{A} \cap Y$.

Exercise 3.22: (Metric topology and subspace)

Suppose (X, d) is a metric space. Given any $A \subset X$, show that d restricts to a metric on A. Show that the subspace topology on any $A \subset X$ is the same as the metric topology for the induced metric space (A, d).

3.5 Continuous function

Definition 3.23: (Continuous function)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *continuous* if $f^{-1}(U) \in \mathcal{T}_X$ for any $U \in \mathcal{T}_Y$ (i.e., pre-image of open sets are open).

Exercise 3.24: (Pre-image of closed set)

Show that $f : X \to Y$ is continuous if and only if preimage of closed sets of Y is closed in X.

Exercise 3.25: (Continuity of the identity)

Suppose X is equipped given topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

Definition 3.26: (Open map)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *open* if $f(U) \in \mathcal{T}_Y$ for any $U \in \mathcal{T}_X$ (i.e, image of opens sets are open).

Exercise 3.27: (Openness of the identity)

Suppose X is equipped given topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ if and only if $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is open.

Exercise 3.28: (Openness of bijection)

Suppose $f : X \to Y$ is a bijection. Show that f is open if and only if f^{-1} is continuous.

Definition 3.29: (Homeomorphism)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be a *homeomorphism* if the following holds.

a) f is bijective, with inverse $f^{-1} : Y \to X$.

b) f is continuous.

c) f is open (or equivalently, f^{-1} is continuous).

Exercise 3.30: (Continuous bijective map)

For $0 \leq t < 1$, consider $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Check that $f : [0, 1) \to \mathbb{R}^2$ is a continuous, injective map. Draw the image. Is it a homeomorphism onto the image (with the corresponding subspace topologies)?

Caution 3.31: (Invariance of domain)

In general, a continuous bijection need not be a homeomorphism. However, there is a special situation known as the *Invariance of domain*. Suppose $U \subset \mathbb{R}^n$ is an open set. Consider a continuous injective map $f : U \to \mathbb{R}^n$. Denote $V := f(U)$. Clearly, $f : U \to V$ is a continuous bijection.

It is a very important theorem in topology that states : V is open and $f : U \to V$ is a homeomorphism.

Definition 3.32: (Closed map)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *closed* if $f(C)$ is closed in Y for any closed set $C \subset X$.

Exercise 3.33: (Open and closed map)

Give examples of continuous maps which are :

a) open, but not closed,

b) closed, but not open,

c) neither open nor closed,

d) both open and closed.

Hint

Consider $f_1(x, y) = x$, $f_2(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$, $f_3(x) = \sin(x)$, and $f_4(x) = x$.

Exercise 3.34: (Continuity is local)

Suppose $X = \bigcup U_\alpha$, for some open sets U_α. Show that $f : X \to Y$ is continuous if and only if $f|_{U_\alpha} \to Y$ is continuous for all α.

Theorem 3.35: (Pasting lemma)

Suppose $X = A \cup B$, for some closed sets $A, B \subset X$. Let $f : A \to Y, g : B \to Y$ be given continuous maps, such that $f(x) = g(x)$ for any $x \in A \cap B$. Then, there exists a (unique) continuous map $h : X \to Y$ such that $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B. \end{cases}$

Proof

Clearly, h is a well-defined function, and it is uniquely defined. We show that h is continuous. Let $C \subset Y$ be a closed set. Then,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Now, $f^{-1}(C) \subset A$ and $g^{-1}(C) \subset B$ are closed sets (in the subspace topology). But then they are closed in X, since A, B are closed. Then, $h^{-1}(C)$ is closed. Since C was arbitrary, we have h is continuous. □

Exercise 3.36: (Pasting lemma for finite collection)

Suppose $X = \bigcup_{i=1}^{n} C_i$ for some closed sets $C_i \subset X$. Let $f_i : C_i \to Y$ be continuous functions such that

$$f_i(x) = f_j(x), \quad x \in C_i \cap C_j, \quad 1 \le i < j \le n.$$

Show that there exists a (unique) continuous function $h : X \to Y$ such that $h(x) = f_i(x)$ whenever $x \in C_i$.

Caution 3.37: (Pasting lemma for infinite collection)

Pasting lemma need not hold true for infinite collection! Consider X to be the integers $\mathbb{Z}$ equipped with the cofinite topology (i.e., open sets are either $\emptyset$ or complements of finite subsets). Check that $\{n\} \subset X$ is closed, and the inclusion map $\iota : X \hookrightarrow \mathbb{R}$ is continuous on each $\{n\}$. Finally, check that ι is not continuous itself.