Topology Course Notes (KSM1C03)

Day 2 : 13th August, 2025

`metric space -- topological space -- basis -- subbasis`

2.1 Metric Spaces

Definition 2.1: (Metric space)

Given a set X, a *metric* on it is a map $d : X \times X \to [0, \infty)$ such that the following holds.

1) a. $d(x, x) = 0$ for all $x \in X$.

   b. If $x \neq y \in X$, then $d(x, y) > 0$.

2) $d(x, y) = d(y, x)$ for all $x, y \in X$

3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The tuple (X, d) is called a metric space. The open ball of radius r, centered at some $x \in X$ is denoted as

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}.$$

Similarly, the closed ball is defined as

$$\bar{B}_d(x, r) := \{y \in X \mid d(x, y) \leq r\}.$$

Definition 2.2: (Open set in metric space)

Given a metric space (X, d), a set $U \subset X$ is called open if

for all $x \in X$, there exists some $r > 0$, such that $B_d(x, r) \subset U$.

Exercise 2.3: (Properties of open sets)

From the definition, verify the following.

i) $\emptyset$ and X are open sets.

ii) Given any collection $\{U_\alpha \subset X\}$ of open sets, the union $\bigcup U_\alpha$ is open in X.

iii) Given a finite collection $\{U_1, \ldots, U_k\}$ of open sets, the intersection $\bigcap_{i=1}^k U_i$ is open in X.

Remark 2.4: (Which properties of metric are needed?!)

You should need $1a$ to show that $x \in B_d(x, r)$, and hence, X is open. You should need 3 to show that

$$B_d(x, \min\{r_1, r_2\}) \subset B_d(x, r_1) \cap B_d(x, r_2),$$

which is needed for the finite intersection.
In particular, $1b$ and 2 are not needed to verify the properties of open sets. Indeed, such general "metric" exists, known as pseud-metric and asymmetric metric.

2.2 Topological Spaces

Definition 2.5: (Topology)

Given a set X, a *topology* on X is a collection $\mathcal{T}$ of subsets of X (i.e., $\mathcal{T} \subset \mathcal{P}(X)$), such that the following holds.

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any collection of elements $U_\alpha \in \mathcal{T}$ with $\alpha \in \mathcal{I}$, an indexing set, we have $\bigcup_{\alpha \in \mathcal{I}} U_\alpha \in \mathcal{T}$.

c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite collection of elements $U_1, \ldots, U_n \in \mathcal{T}$, we have $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The tuple $(X, \mathcal{T})$ is called a topological space.

Example 2.6

Given any set X we always have two standard topologies on it.

a) **(Discrete Topology)** $\mathcal{T}_0 = \mathcal{P}(X)$.

b) **(Indiscrete Topology)** $\mathcal{T}_1 = \{\emptyset, X\}$.

They are distinct whenever X has at least 2 points.

Exercise 2.7

Given any set X, verify that both the discrete and the indiscrete topologies are indeed topologies, that is, check that they satisfy the axioms.

Definition 2.8: (Metric topology)

Given a metric space (X, d), the collection of open sets in X form a topology, called the *metric topology* (or the *topology induced by the metric*).

Exercise 2.9: (Metric inducing discrete and indiscrete topology)

Given a set X, can you give a metric on it such that the induced topology on X is the discrete topology? Can you do the same for indiscrete topology?

Exercise 2.10: (Topologies on 3-point set)

Suppose $X = \{a, b, c\}$. Note that

$$|\mathcal{P}(\mathcal{P}(X))| = 2^{|\mathcal{P}(X)|} = 2^{2^{|X|}} = 2^{2^3} = 256.$$

Thus, there are 256 possible collections of subsets of X. How many of them are topologies? How many are distinct if you are allowed to permute the elements $\{a, b, c\}$?

Hint

The answers should be 29 and 9.

Definition 2.11: (Open and closed sets)

Given a topological space $(X, \mathcal{T})$, a subset $U \subset X$ is called an *open set* if $U \in \mathcal{T}$, and a subset $C \subset X$ is called a *closed set* if $X \setminus C \in \mathcal{T}$ (i.e., if $X \setminus C$ is open).

Caution 2.12

Given $(X, \mathcal{T})$, a subset can be both open and closed! Think about the discrete topology. Such subsets are sometimes called *clopen sets*.

Exercise 2.13: (Topology defined by closed sets)

Given X, suppose $\mathcal{C} \subset \mathcal{P}(X)$ is a collection of subsets that satisfy the following.

a) $\emptyset \in \mathcal{C},\ X \in \mathcal{C}$.

b) $\mathcal{C}$ is closed under arbitrary intersections.

c) $\mathcal{C}$ is closed under finite unions.

Define the collection,

$$\mathcal{T} := \{U \subset X \mid X \setminus U \in \mathcal{C}\}.$$

Prove that $\mathcal{T}$ is a topology on X.

Exercise 2.14

On the real line $\mathbb{R}$, consider the collection of subsets

$$\mathcal{T}_{\leftarrow} := \{\emptyset, \mathbb{R}\} \bigcup \{(-\infty, a) \mid a \in \mathbb{R}\}.$$

Show that $\mathcal{T}_{\leftarrow}$ is a topology on $\mathbb{R}$.

2.3 Basis of a topology

Definition 2.15: (Basis of a topology)

Given a topological space $(X, \mathcal{T})$, a *basis* for it is a sub-collection $\mathcal{B} \subset \mathcal{T}$ of open sets such that every open set $U \in \mathcal{T}$ can be written as the union of some elements of $\mathcal{B}$.

Example 2.16: (Usual topology on $\mathbb{R}$)

The collection of all open intervals $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$ is a basis for the usual topology on the real line $\mathbb{R}$.

Proposition 2.17: (Necessary condition for basis)

Suppose $(X, \mathcal{T})$ is a topological space, and consider a basis $\mathcal{B} \subset \mathcal{T}$. Then, the following holds.

(B1) For any $x \in X$, there exists some $U \in \mathcal{B}$ such that $x \in U$.

(B2) For any $U, V \in \mathcal{B}$ and any element $x \in U \cap V$, there exists some $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

Proof

Suppose $\mathcal{B}$ is a basis of $(X, \mathcal{T})$. Since X is open in X, we have $X = \bigcup_{O \in \mathcal{B}} O$, which implies (B1). Now, for any $U, V \in \mathcal{B}$, we have $U \cap V$ is open as well. Thus, $U \cap V$ is the union of some elements of $\mathcal{B}$, which implies (B2). □

Example 2.18

Consider the collection

$$\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}.$$

This is a subcollection of open sets of $\mathbb{R}$ (in the usual topology), and moreover, $\mathcal{B}$ satisfies both $B1$ and $B2$ (Check!). But $\mathcal{B}$ is *not* a basis for the usual topology on $\mathbb{R}$. Thus, $B1$ and $B2$ is not a sufficient condition for $\mathcal{B}$ to be a basis.

Exercise 2.19: (Topology generated by a basis)

Suppose $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X satisfying (B1) and (B2). Consider $\mathcal{T}$ to be the collection of all possible unions of elements of $\mathcal{B}$. Show that $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for it.

Exercise 2.20: (Basis for metric topology)

Suppose (X, d) is a metric space. Consider the collection

$$\mathcal{B} := \{B_r(x) \mid x \in X,\ r > 0\},$$

where $B_r(x) := \{y \mid d(x, y) < r\}$ is the ball of radius r, centered at x. Show that $\mathcal{B}$ is a basis for a topology on X, known as the *metric topology* induced by the metric d.

Exercise 2.21: (Closed discs generate discrete topology)

Let (X, d) be a metric space, and $\bar{B}_r(x) = \{y \in X \mid d(x, y) \leq r\}$ be the *closed* ball of radius r centered at x. Show that the collection

$$\mathcal{B} := \left\{\bar{B}_r(x) \,\middle|\, x \in X, r \geq 0\right\}$$

is a basis for the discrete topology on X.

Exercise 2.22: (Usual topology on $\mathbb{R}^2$)

Consider the following collections of subsets of the plane $\mathbb{R}^2$.

a) $\mathcal{B}_1$ be the collection of all open discs with all possible radii and center at any point.

b) $\mathcal{B}_2$ be the collection of all open discs with radius less than 1, and center at any point.

c) $\mathcal{B}_3$ be the collection of all open squares (i.e, only the insides, not the boundary) with sides parallel to the two axes.

Show that all three are bases for the usual topology on $\mathbb{R}^2$.

Hint

Draw pictures!

2.4 Subbasis of a topology

Definition 2.23: (Subbasis of a topology)

Given a topological space $(X, \mathcal{T})$, a *subbasis* is a collection of subsets $\mathcal{S} \subset \mathcal{T}$ such that $\mathcal{T}$ is the smallest topology on X containing $\mathcal{S}$.

Proposition 2.24: (Topology generated by subbasis)

Let X be a set, and $\mathcal{S}$ be any collection of subsets of X (i.e, $\mathcal{S} \subset \mathcal{P}(X)$). Then, $\mathcal{S}$ is a subbasis for a (unique) topology on X (called the *topology generated $\mathcal{S}$*).

Proof

Consider the collection

$$\mathfrak{T} := \{\mathcal{T} \subset \mathcal{P}(X) \mid \mathcal{T} \text{ is a topology and } \mathcal{S} \subset \mathcal{T}\}.$$

Note that it is a nonempty collection, as $\mathcal{P}(X) \in \mathfrak{T}$. Denote $\mathcal{T}_0 = \bigcap_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}$. Then $\mathcal{T}_0$ is a topology, and by definition, it is the smallest one containing $\mathcal{S}$. □

Explicitly, an open set of the topology generated by a subbasis $\mathcal{S}$ can be (non-uniquely) written as an arbitrary union of finite intersections of elements of $\mathcal{S}$.

Exercise 2.25: (Trivial subbases)

Given any set X, figure out the topologies generated by the following sub-bases :

$$\mathcal{S}_1 = \emptyset, \quad \mathcal{S}_2 = \{\emptyset\}, \quad \mathcal{S}_3 = \{X\}, \quad \mathcal{S}_4 = \{\emptyset, X\}.$$

Exercise 2.26

Given the plane $\mathbb{R}^2$ consider the collection

$$\mathcal{S} := \left\{ B_1(x) \,\middle|\, x \in \mathbb{R}^2 \right\},$$

where $B_1(x)$ is the unit open disc centered at x. Show that

a) $\mathcal{S}$ is not a basis for any topology on $\mathbb{R}^2$, but

b) the topology generated by $\mathcal{S}$ is the usual metric topology.

Hint

Place 4 unit discs with centers at the four corners of a square, with side length strictly less than 2. Look at the intersection!

2.5 Fine and coarse topology

Definition 2.27: (Fine and coarse topology)

Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on a set X, we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is said to be *coarser* than $\mathcal{T}_1$) if $\mathcal{T}_1 \supset \mathcal{T}_2$.

Caution 2.28

One way to remember the terminology is to think of each open set as small pebbles. If you crush each pebble in to finer pebbles, then you get more of it! Thus, the finer collection is larger (has more open sets), and the coarser collection is smaller (has less number of open sets).

Exercise 2.29

Check that the discrete topology on a set X is the finest, i.e., finer than any other topology that can be given on X. Dually, the indiscrete topology is the coarsest topology.

Caution 2.30

Not all topologies on a set are comparable to each other! Can you construct such examples on $\{a, b, c\}$?

Exercise 2.31

Show that the lower limit topology $\mathbb{R}_l$ is stictly finer than the usual topology on $\mathbb{R}$.