Topology Course Notes (KSM1C03)

Day 1 : 12^{th} August, 2025

`basic set theory -- power set -- product of sets -- equivalence relation -- order relation`

1.1 Power set

Given a set X, the *power set* is defined as

$$\mathcal{P}(X) := \{A \mid A \subset X\}.$$

Exercise 1.1

If X is a finite set, prove via induction that $|\mathcal{P}(X)| = 2^{|X|}$, where $|\cdot|$ denotes the cardinality.

Exercise 1.2

For any arbitrary set X, prove that there exists a natural bijection of $\mathcal{P}(X)$ with the set

$$\mathcal{F} := \{f : X \to \{0,1\}\}$$

of all functions from X to the 2-point set $\{0,1\}$.

Hint

How many functions $\{a,b,c\} \to \{0,1\}$ can you define? Look at their inverse images.

Given two sets X, Y denote the set of all functions from X to Y as

$$Y^X := \{f : X \to Y\}.$$

Exercise 1.3

If X and Y are finite sets, then show that $\left|Y^X\right| = |Y|^{|X|}$. Use this to show $|\mathcal{P}(X)| = 2^{|X|}$.

Exercise 1.4: (Set exponential law)

Given three sets X, Y, Z, prove that there is a natural bijection

$$\left(Z^Y\right)^X = Z^{Y \times X}$$

Hint

Write down what the elements look like. Can you see the pattern? This bijection is also known as *Currying*.

1.2 Arbitrary union and intersection

Suppose $\mathcal{A}$ is a collection of sets. Then, we have the *union*

$$\bigcup_{X \in \mathcal{A}} X := \{x \mid x \in X \text{ for } \textbf{some } X \in \mathcal{A}\},$$

and the *intersection*

$$\bigcap_{X \in \mathcal{A}} X := \{x \mid x \in X \text{ for } \textbf{all } X \in \mathcal{A}\}.$$

Exercise 1.5: (Empty union)

Suppose we have an *empty* collection $\mathcal{A}$ of sets. From the definition, prove that

$$\bigcup_{X \in \mathcal{A}} X = \emptyset.$$

Exercise 1.6: (Empty intersection)

Suppose $\mathcal{A}$ is a *nonempty* subset of the power set of some fixed set X. Show that

$$\bigcap_{A \in \mathcal{A}} = \{x \in X \mid x \in A \text{ for all } A \in \mathcal{A}\}.$$

If $\mathcal{A} \subset \mathcal{P}(X)$ is the *empty* collection, justify

$$\bigcap_{A \in \mathcal{A}} A = X$$

1.3 Cartesian product

Given two sets A, B, their *Cartesian product* (or simply, *product*) is defined as the set

$$A \times B := \{(a, b) \mid a \in A, \quad b \in B\}$$

of ordered pairs. We have the two *projections*

$$\begin{aligned} \pi_A : A \times B &\to A \\ (a, b) &\mapsto a, \end{aligned} \quad \text{and} \quad \begin{aligned} \pi_B : A \times B &\to B \\ (a, b) &\mapsto b. \end{aligned}$$

Exercise 1.7

Justify $A \times \emptyset = \emptyset$, where $\emptyset$ is the empty set.

Remark 1.8: (A different product?)

Suppose A, B are given. Consider the set

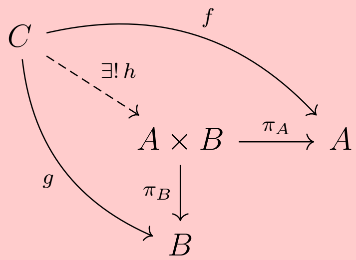
$$C = \{(a,b,a) \mid a \in A, \quad b \in B\}.$$

Clearly there is a natural bijection between C and $A \times B$. Also, we have maps $\pi_A : C \to A$ and $\pi_B : C \to B$.

Exercise 1.9: (Universal property of the product)

Suppose A, B are given sets, and $\pi_A : A \times B \to A, \pi_B : A \times B \to B$ be the projections.

a) Show that given any set C, and functions $f : C \to A, g : C \to B$, there exists a *unique* function $h : C \to A \times B$ such that the diagram commutes.

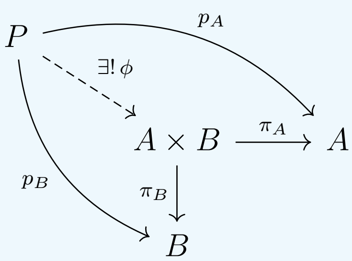


b) Suppose we are given a set P, along with two functions $p_A : P \to A$ and $p_B \to B$, which satisfies the following property : given any set C, and functions $f : C \to A, \ g : C \to B$, there exists a *unique* function $h : C \to P$ satisfying $f = p_A \circ h, \ g = p_B \circ h$.

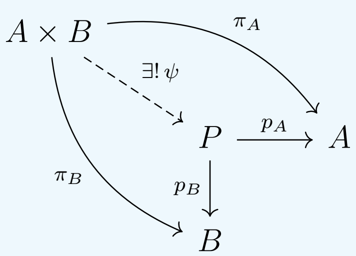
Show that the exists a bijection from $\psi : A \times B \to P$, such that $p_A \circ \psi = \pi_A$ and $p_B \circ \psi = \pi_B$.

Hint

Look at the diagrams



and



Can you show that $\phi \circ \psi = \mathrm{Id}_{A \times B}$ and $\psi \circ \phi = \mathrm{Id}_P$? The uniqueness should be useful.

1.4 Equivalence relation

Definition 1.10: (Relation)

Given a set X, a *relation* on it is a subset $\mathcal{R} \subset X \times X$. We say $\mathcal{R}$ is an *equivalence relation* if the following holds.

a) **(Reflexive)** For each $x \in X$ we have $(x, x) \in \mathcal{R}$.

b) **(Symmetric)** If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.

c) **(Transitive)** If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

For any $x \in X$, the *equivalence class* (with respect to the equivalence relation $\mathcal{R}$) is defined as the set

$$[x] \coloneqq \{y \in X \mid (x, y) \in \mathcal{R}\}.$$

We shall denote $x \sim_{\mathcal{R}} y$ (sometimes also denoted $x\mathcal{R}y$, or simply $x \sim y$) whenever $(x, y) \in \mathcal{R}$. The collection of equivalence classes are sometimes denoted as $X/_\sim$.

Exercise 1.11

Given an equivalence relation $\mathcal{R}$ on X, check that any two equivalence classes are either disjoint or equal (i.e., they cannot have nontrivial intersection).

Exercise 1.12

Suppose X is a given set, and $A \subset X$ is a nonempty subset. Define the relation $\mathcal{R} \subset X \times X$ as follows.

$$\mathcal{R} \coloneqq \{(x, x) \mid x \in X \setminus A\} \bigcup \{(a, b) \mid a, b \in A\}.$$

a) Check that $\mathcal{R}$ is an equivalence relation.

b) Identify the equivalence classes. We shall denote the collection of equivalence classes as X/A.

c) What is X/X ?

Exercise 1.13

Suppose G is a group and H is a subgroup. Define a relation

$$\mathcal{C} \coloneqq \left\{(g_1, g_2) \,\middle|\, g_1^{-1} g_2 \in H\right\} \subset G \times G.$$

a) Show that $\mathcal{C}$ is an equivalence relation.

b) Identify the equivalence classes G/H.

Hint

Recall the definition of cosets.

Definition 1.14: Partition

Given a set X, a *partition of X* is a collection of subsets $X_\alpha \subset X$ for some indexing set $\alpha \in \mathcal{I}$, such that the following holds.

- $X_\alpha \cap X_\beta = \emptyset$ for any $\alpha, \beta \in \mathcal{I}$ with $\alpha \neq \beta$.
- $X = \bigcup_{\alpha \in \mathcal{I}} X_\alpha$.

Exercise 1.15: (Partitions and equivalence relations)

Given an equivalence relation $\mathcal{R}$ on a set X, show that the collection of equivalence classes is a partition of X. Conversely, given any partition of X, show that there exists a unique equivalence relation which gives that partition.

1.5 Order relation

Definition 1.16: (Linear order)

A relation $\mathcal{O} \subset X \times X$ on X is called an *order relation* (also known as *linear order* or *simple order*) if the following holds.

a) **(Non-reflexive)** $(x, x) \notin \mathcal{O}$ for all $x \in X$.

b) **(Transitive)** If $(x, y) \in \mathcal{O}$ and $(y, z) \in \mathcal{O}$, then $(x, z) \in \mathcal{O}$.

c) **(Comparable)** For $x, y \in X$ with $x \neq y$, either $(x, y) \in \mathcal{O}$ or $(y, x) \in \mathcal{O}$.

We shall denote $x <_{\mathcal{O}} y$ (or even simply $x < y$) whenever $(x, y) \in \mathcal{O}$. If either $x <_{\mathcal{O}} y$ or $x = y$ holds, then we shall denote $x \leq_{\mathcal{O}} y$ (or $x \leq y$). Given $x, y \in X$, we have the interval

$$(x, y) := \{z \in X \mid x < z \text{ and } z < x\}.$$

Exercise 1.17

Given an ordered set $(X, <)$, define the intervals $[x, y], [x, y), (x, y]$ for some $x, y \in X$. What happens when $x = y$?

Definition 1.18: (Order preserving function)

Given two ordered set $(X_1, <_1)$ and $(X_2, <_2)$, a function $f : X_1 \to X_2$ is said to *order preserving* if

$$x <_1 y \Rightarrow f(x) <_2 f(y), \quad \forall\, x, y \in X_1.$$

Definition 1.19: (Total order)

A relation $\mathcal{O} \subset X \times X$ on a set X is called a *total order* if the following holds.

a) **(Reflexive)** $(x, x) \in \mathcal{O}$ for all $x \in X$.

b) **(Transitive)** If $(x, y) \in \mathcal{O}$ and $(y, z) \in \mathcal{O}$, then $(x, z) \in \mathcal{O}$.

c) **(Total)** For $x, y \in X$ either $(x, y) \in \mathcal{O}$ or $(y, x) \in \mathcal{O}$

d) **(Antisymmetric)** If $(x, y) \in \mathcal{O}$ and $(y, x) \in \mathcal{O}$, then $x = y$.

We shall denote $x \leq_{\mathcal{O}} y$ (or even simply $x \leq y$) whenever $(x, y) \in \mathcal{O}$.

Definition 1.20: (Dictionary order)

Given X, Y two totally ordered sets the *dictionary order* (or *lexicographic order*) on the product $X \times Y$ is defined as

$$(x_1, y_1) < (x_2, y_2) \text{ if and only if } \{x_1 < x_2\} \text{ or } \{x_1 = x_2, \text{and} y_1 < y_2\},$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Exercise 1.21

Let X, Y be totally ordered sets.

a) Check that the dictionary order on $X \times Y$ is indeed a total ordering.

b) Check that the projection maps $\pi_X \to X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are order preserving maps.

c) Suppose Z is another totally ordered set. Let $f : Z \to X$ and $g : Z \to Y$ be two order preserving maps. Show that there exists a unique order preserving map $h : Z \to X \times Y$ such that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$.

d) Let us define a new relation $(x_1, y_1) \preceq (x_2, y_2)$ if and only $x_1 \leq x_2$ and $y_1 \leq y_2$. Is $\preceq$ a total order on $X \times Y$?